Journal of Sound and Vibration (2001) **246**(5), 927–941
doi:10.1006/jsvi.2001.3620, available online at http://www.idealibrary.com on IDEAL®

LETTERS TO THE EDITOR

FREQUENCIES AND MODE SHAPES FOR FINITE LENGTH CYLINDERS

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(*Received* 1 *August* 2000, *and in final form* 8 *January* 2001)

1. INTRODUCTION

A recent study reported by Leissa and So [1] establishes accurate vibration frequencies for finite length cylinders with free–free boundary conditions. In their work they also report some results for finite length cylinders with fixed–free boundary conditions. It is of interest to note that they comment on the absence of published results for vibration of finite length cylinders with boundary conditions other than free–free. For instance, an absence of vibration studies for fixed–free finite length cylinders. Additionally, So and Leissa [2] in a later report, offer comments on studies of free–free finite length cylinders that they classified as thick hollow cylinders. Additional pertinent references are given in reference [2]. It is noteworthy that Hutchinson [3] gave results for an encased cylinder and that study represents several combinations of boundary conditions for a finite length cylinder. The analysis to be reported herein will dwell upon a variety of boundary conditions for isotropic finite length cylinders and then continue with finite length cylinders with hexagonal material properties. Hexagonal material properties for beryllium [4, 5] are used in this study.

There are some free vibration studies reported that concern finite length cylinders with anisotropic material properties. Lusher and Hardy [6] gave results for frequency and mode shapes for a free–free cylinder with material properties of sapphire. Sapphire is of crystal class $\bar{3}$m and would not be suitable for an axisymmetric analysis. However, Lusher and Hardy [6] argue, and justifiably, that the stiffness constant C_{14} is small compared with the remaining material constants and can be neglected. In that case the material qualifies as an axisymmetric hexagonal class. Heyliger [7] reports results for an anisotropic finite length cylinder and uses the material properties given in [6]. It follows that he was able to verify the work in reference [6] using a hexagonal form of sapphire. In later work, Heyliger and Jilani [8] gave useful results for isotropic solid and hollow cylinders and extended the previous analysis to include higher circumferential wave numbers. They also gave the three- dimensional displacement equations of motion for an orthotropic cylinder.

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2. GOVERNING EQUATIONS AND FINITE ELEMENT MODEL

The important equations are the strain–displacement equations in cylindrical (r, θ, z) co-ordinates and can be written in single subscript form as

$$\begin{aligned} &\varepsilon_{rr} = \varepsilon_1 = \frac{\partial u}{\partial r}, && \varepsilon_{\theta\theta} = \varepsilon_2 = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}, \\ &\varepsilon_{zz} = \varepsilon_3 = \frac{\partial w}{\partial z}, && \varepsilon_{\theta z} = \varepsilon_4 = \frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta}, \\ &\varepsilon_{rz} = \varepsilon_5 = \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}, && \varepsilon_{r\theta} = \varepsilon_6 = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}, \end{aligned} \tag{1}$$

where u, v and w are the displacements in the r, θ, z directions respectively. The formulation is three dimensional and requires the complete matrix of stiffness material constants. The stress–strain equations are

$$\sigma_k = C_{kl}\varepsilon_l, \tag{2}$$

where the C_{kl} for hexagonal materials can be written as

$$C_{kl} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & & & \\ C_{12} & C_{11} & C_{13} & & & \\ C_{13} & C_{13} & C_{33} & & & \\ & & & C_{44} & & \\ & & & & C_{44} & \\ & & & & & C_{66} \end{bmatrix} \tag{3}$$

where $C_{66} = (C_{11} - C_{12})/2$. The isotropic material matrix is recovered by identifying $C_{33} = C_{11}$, $C_{13} = C_{12}$ and $C_{44} = C_{66}$. A hexagonal material such as beryllium (Be) is non-piezoelectric and, according to Nye [9], is further classified as 6/mmm.

The global finite element formulation can be derived using the Rayleigh–Ritz method or the Galerkin method. The development is standard and can be found in numerous textbooks [10]. The displacements, u, v, and w will be referred to symbolically as u and are assumed in terms of nodal point variables,

$$u = [N]\{u\}, \tag{4}$$

where $\{u\}$ is the displacement vector that defines the local element nodal point displacements and $[N]$ is a suitable set of assumed shape functions. The shape functions can be formulated directly in axisymmetric (r, z) co-ordinates or can be formulated as isoparametric shape functions. Both formulations have been used and give identical results. The advantage of the isoparametric element is that the analysis is not limited to a right circular cylinder. In this study nine-node Lagrangian shape functions were used and written in (r, z) co-ordinates. The local finite element equation for dynamic elasticity for cylinders with zero surface traction can be written as

$$\int_V [B]^{\mathrm{T}}[C][B]\{u\}\,\mathrm{d}V - \int_V [N]^{\mathrm{T}}[\rho][N]\frac{\partial^2\{u\}}{\partial t^2}\,\mathrm{d}V = 0, \tag{5}$$

where $[C]$ is a form of equation (3), $[\rho]$ is the density matrix and $[B]$ is further defined as

$$[B] = [L]\,[N]. \tag{6}$$

The operator matrix $[L]$ is defined according to the strain–displacement equations given by equation (1):

$$[L] = \begin{bmatrix} \dfrac{\partial}{\partial r} & 0 & 0 \\ \dfrac{1}{r} & \dfrac{1}{r}\dfrac{\partial}{\partial \theta} & 0 \\ 0 & 0 & \dfrac{\partial}{\partial z} \\ 0 & \dfrac{\partial}{\partial z} & \dfrac{1}{r}\dfrac{\partial}{\partial \theta} \\ \dfrac{\partial}{\partial z} & 0 & \dfrac{\partial}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial r} - \dfrac{1}{r} & 0 \end{bmatrix}. \tag{7}$$

The three-dimensional spatial formulation is reduced to two dimensions by assuming periodic solutions that will separate θ and time dependence from r and z. However, each finite element node still has three degrees of freedom. The following assumptions are in order [1]:

$$u(r, \theta, z, t) = U(r, z) \cos \mathrm{m}\theta \cos \omega t, \tag{8}$$

$$v(r, \theta, z, t) = V(r, z) \sin \mathrm{m}\theta \cos \omega t, \tag{9}$$

$$w(r, \theta, z, t) = W(r, z) \cos \mathrm{m}\theta \cos \omega t, \tag{10}$$

where m is an integer that is the circumferential wave number and ω is the circular frequency. Substituting into equation (6) gives the $[B]$ matrix in terms of shape functions and the circumferential wave number m. The development of the finite element model is still quite general and would be applicable for any set of assumed shape functions. The $[B]$ matrix becomes

$$[B] = \begin{bmatrix} \dfrac{\partial N_1}{\partial r} & 0 & 0 & \cdots\cdots\cdots \\ \dfrac{N_1}{r} & \dfrac{mN_1}{r} & 0 & \cdots\cdots\cdots \\ 0 & 0 & \dfrac{\partial N_1}{\partial z} & \cdots\cdots\cdots \\ 0 & \dfrac{\partial N_1}{\partial z} & -\dfrac{mN_1}{r} & \cdots\cdots\cdots \\ \dfrac{\partial N_1}{\partial z} & 0 & \dfrac{\partial N_1}{\partial r} & \cdots\cdots\cdots \\ -\dfrac{mN_1}{r} & \dfrac{\partial N_1}{\partial r} - \dfrac{N_1}{r} & 0 & \cdots\cdots\cdots \end{bmatrix}. \tag{11}$$

The nine-node element would require the $[B]$ matrix to be 6×27. The three columns of equation (11) would be repeated for N_2 through N_9. A consistent mass matrix is developed following the formulation given by equation (5). The matrix $[N]$ of equation (5) would be 3×27 in order to accommodate three degrees of freedom. It follows that $[\rho]$ would be a 3×3 diagonal matrix defined as

$$[\rho]=\begin{bmatrix}\rho & 0 & 0\\ 0 & \rho & 0\\ 0 & 0 & \rho\end{bmatrix}. \tag{12}$$

The mass matrix becomes

$$[M]=\int_V [N]^{\mathrm{T}}[\rho][N]\,\mathrm{d}V. \tag{13}$$

Equation (5) becomes a standard eigenvalue problem that can be written as

$$[K]\{u\}-\omega^2[M]\{u\}=0. \tag{14}$$

Two basic global models were used to model vibrating finite length cylinders: a 40-element model with 187 nodes and 561 total degrees of freedom and a 50-element model with 231 nodes and 693 total degrees of freedom. In every case only the first nine or ten frequencies were of interest and the difference between the results obtained from the two models was negligible. For the most part, the model with fewer degrees of freedom was used.

3. NUMERICAL RESULTS

An axisymmetric solid cylinder is assumed as illustrated in Figure 1 with radius a and height L. Variables are non-dimensionalized with respect to radius a and material constant C_{44}. It follows that non-dimensional co-ordinates, material constants, time and frequency are defined as

$$\bar{r}=r/a,\ \bar{z}=z/a,\ \bar{C}_{k1}=\bar{C}_{k1}/C_{44},\quad \bar{t}=t\sqrt{C_{44}/\rho a^2},\ \Omega=\omega a\sqrt{\rho/C_{44}}. \tag{15}$$

Results for natural frequencies and mode shapes will be given for two sets of boundary conditions, three cylinder heights and two materials. The boundary conditions can be classified as fixed–free and fixed–fixed. The cylinder is fixed at the base $z=0$ and free at $z=L$ and the second case is fixed against motion in the all co-ordinate directions at $z=0$ and L. The cylinder height is assumed to be $L/a=1{\cdot}0$ representing a thick disk, $L/a=2{\cdot}0$ representing a short cylinder ad $L/a=4{\cdot}0$ representing a long cylinder. The materials are given in Table 1 for an isotropic material with the Poisson ratio $\nu=0{\cdot}3$ and beryllium. The material properties that have been used for this study represent contrasting ratios of C_{11} compared with C_{12} and C_{44}. Table 1 illustrates the relative magnitudes of the material constants in non-dimensional format.

Frequencies for fixed–free isotropic cylinders are given in Table 2 and compare favorably with upper bound results given by Leissa and So [1] and So [11]. The frequencies of Table 2 validate the finite element analysis as well as extend the existing results to include frequencies for beryllium cylinders with fixed–free boundary conditions. Representative modes shapes for fixed–free cylinders are shown in Figures 2–4.

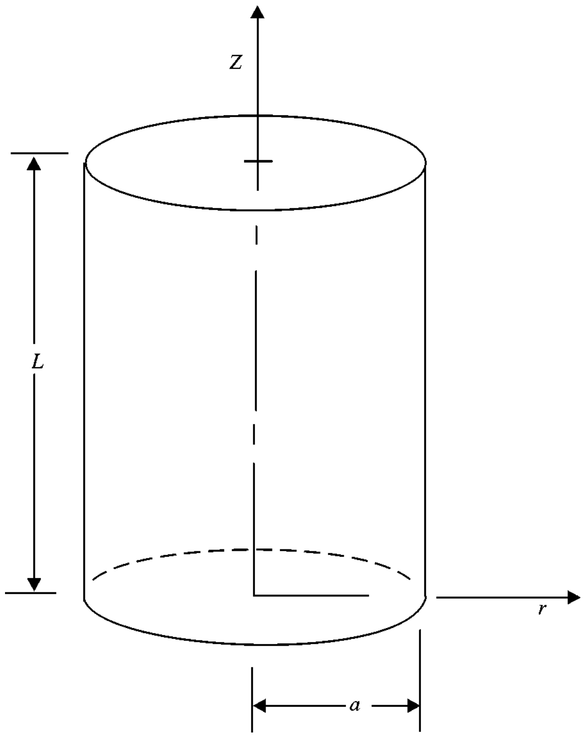

Figure 1. Finite length cylinder of radius a and length L.

TABLE 1

Elastic constants for Beryllium (Be) from reference [4, 5]; *multiply by* 10^{10} N/m^2

	C_{11}	C_{33}	C_{12}	C_{13}	C_{44}	C_{66}
Be	29·23	33·64	2·67	1·40	16·25	13·28
Be†	1·799	2·070	0·164	0·086	1·00	0·817
Isotropic‡	3·5	3·5	1·5	1·5	1·0	1·0

†non-dimensional C_{1j}/C_{44};
‡non-dimensional, $\nu = 0{\cdot}3$.

Torsional frequencies and longitudinal/radial frequencies can be separated when the circumferential wave number $m = 0$. Equations (11) and (1), when $m = 0$, illustrate that shear strains that involve θ, $\varepsilon_{\theta z}$ and $\varepsilon_{r\theta}$, are uncoupled from the remaining strains. Pure torsional modes can be computed by setting all U and W displacements equal to zero as boundary conditions in the finite element analysis. Similarly, setting V to zero produces pure longitudinal/radial frequencies and mode shapes when $m = 0$.

The mode shapes and frequencies of Figures 2–4 correspond to an isotropic material. Modes shapes for beryllium cylinders with fixed–free boundary conditions show similar behavior as reported by Chua [12]. It turns out that the seventh and eighth mode shapes of Figure 2 would appear as the eighth and seventh modes of Table 2 for a beryllium cylinder

TABLE 2

Frequencies Ω for solid cylinders with fixed–free boundary conditions

		Isotropic $\nu = 0{\cdot}3$						Beryllium		
		L/a						L/a		
m	mode	1	1‡	2	2§	4	4§	1	2	4
0	1	2·558 4	2·558	1·286 3	1·286	0·640 3	0·640	2·225 4	1·128 3	0·564 3
	2	3·317 8	3·316	2·960 5	2·960	1·860 7	1·859	2·498 5	2·359 8	1·691 7
	3	4·039	4·039	3·169	3·169	2·787	2·783	3·295	2·604	2·334
	4	5·527	5·524	4·184	4·182	2·953	2·951	4·835	3·312	2·536
	5	6·551	6·548	4·298	4·297	3·357	3·346	5·903	3·404	2·569
	6	7·234		4·488		3·654		6·658	4·039	2·780
	7	7·472		5·357		3·891		6·773	4·720	2·847
	8	8·709		5·760		3·974		7·033	5·058	3·200
	9	9·045		6·251		4·124		7·743	5·645	3·687
0†	1	1·571 2		0·786 2	0·785	0·393 2	0·393	1·571 2	0·785 2	0·393 2
	2	4·713		2·356 6	2·356	1·178 5	1·178	4·713	2·356 7	1·178 5
	3	5·372		3·929	3·927	1·965 9	1·963	4·902	3·929	1·965
	4	6·972		5·197	5·195	2·755	2·749	6·616	4·710	2·755
	5	7·859		5·510	5·498	3·555	3·544	7·792	5·207	3·555
	6	8·589		5·652		4·372		7·859	5·510	4·372
	7	9·389		6·468		5·153		8·970	6·083	4·660
	8	9·670		7·110		5·215		9·130	7·110	4·791
	9	11·020		7·534		5·271		10·743	7·206	5·042
1	1	1·305 1	1·303	0·506 1	0·506	0·158 1	0·159	1·225 1	0·459 1	0·142 1
	2	2·738 5	2·738	1·445 4	1·444	0·652 4	0·650	2·538 6	1·357 4	0·605 4
	3	3·230 6	3·230	2·592 8	2·588	1·392 6	1·383	2·764 7	2·434 9	1·303 6
	4	4·121	4·117	2·855 9	2·854	1·940 8	1·928	3·704	2·529	1·870 8
	5	5·209	5·204	3·358	3·346	2·396	2·368	4·839	3·060	2·308
	6	5·494		3·635		2·611		5·043	3·259	2·354
	7	5·925		4·049		2·895		5·291	3·767	2·548
	8	6·870		4·638		2·975		6·224	4·293	2·585
	9	6·988		4·880		3·237		6·327	4·559	2·972

2	1	2·435 $\underline{3}$	2·434	2·162 $\underline{5}$	2·162	2·134	2·132	2·205 $\underline{3}$	1·989 $\underline{5}$	1·961 $\underline{9}$
	2	3·305	3·305	2·519 $\underline{7}$	2·518	2·350	2·350	3·038 $\underline{8}$	2·274 $\underline{6}$	2·113
	3	4·531	4·530	3·456	3·449	2·508	2·503	3·878	3·210	2·262
	4	4·899	4·854	3·767	3·760	2·869	2·847	4·512	3·516	2·627
	5	6·233	6·219	4·356	4·347	3·314	3·292	5·786	3·725	3·135
	6	6·392		4·760		3·417		5·968	4·249	3·268
	7	7·006		4·863		3·866		6·408	4·505	3·591
	8	7·387		5·307		4·007		6·582	4·800	3·666
	9	8·170		5·910		4·365		7·230	5·418	3·690
3	1	3·405 $\underline{9}$	3·404	3·260	3·258	3·271	3·253	3·145 $\underline{9}$	3·037	3·038
	2	4·168	4·166	3·699	3·698	3·615	3·164	3·748	3·286	3·225
	3	5·701	5·693	4·298	4·289	3·678	3·674	5·075	3·890	3·286
	4	5·936	5·932	4·658	4·651	3·898	3·878	5·322	4·391	3·503
	5	7·086	7·063	5·377	5·350	4·284	4·168	6·417	4·935	3·887
	6	7·408		5·617		4·369		6·900	5·112	4·233
	7	7·697		6·109		4·742		7·055	5·287	4·425
	8	8·212		6·282		4·831		7·652	5·602	4·582
	9	9·351		6·870		5·255		8·062	6·269	4·906
4	1	4·348	4·347	4·290	4·281	4·335	4·281	4·062	4·016	4·044
	2	5·122	5·120	4·784	4·783	4·724	4·723	4·540	4·219	4·179
	3	6·527	6·517	5·213	5·204	4·774	4·770	5·968	4·658	4·234
	4	7·103	7·093	5·617	5·608	4·935	4·917	6·398	5·243	4·396
	5	7·945	7·918	6·174	6·140	5·232	5·173	7·030	5·797	4·691
	6	8·383		6·427		5·420		7·756	5·963	5·055
	7	8·753		7·200		5·651		7·947	6·442	5·396
	8	9·100		7·468		5·742		8·580	6·632	5·513
	9	10·374		7·760		6·121		9·103	6·917	5·761

†torsional mode;
‡So [11];
§Leissa and So [1].

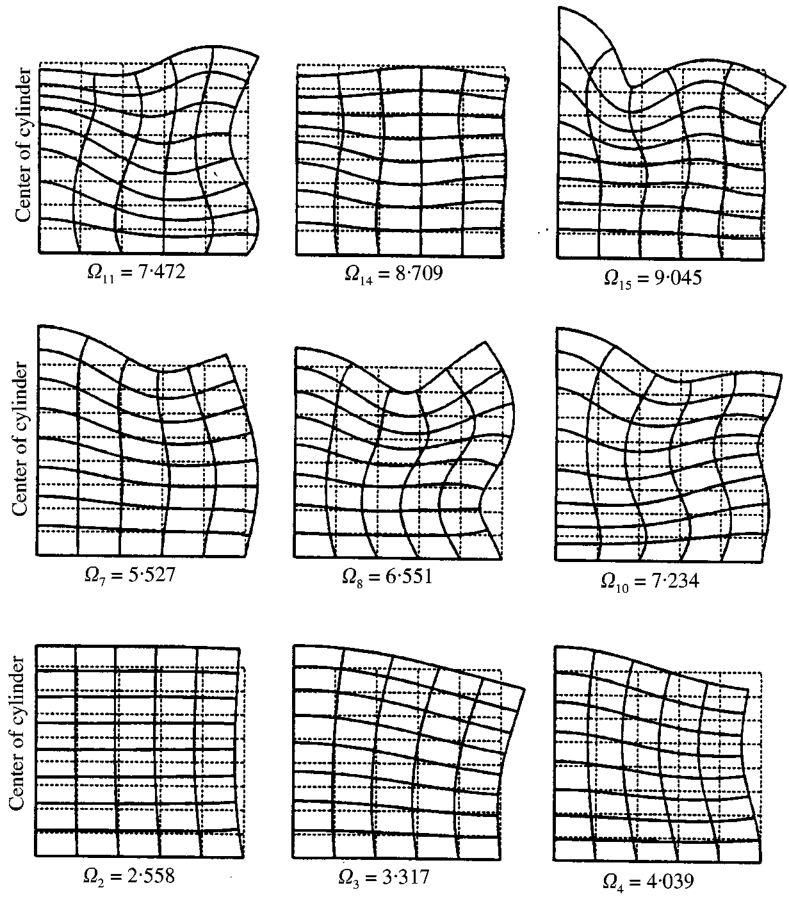

Figure 2. Mode shapes for longitudinal/radial motion of fixed–free cylinders with $L/a = 1$ and isotropic material with $\nu = 0{\cdot}3$ and $m = 0$ corresponding to Table 2.

with $L/a = 1$. Similarly for Figure 3, isotropic and beryllium mode shapes five and six are interchanged with all others being the same for $m = 0$, modes 1–9. Beryllium mode shapes for $L/a = 4$ are the same as in Figure 4 with mode shapes 3 and 4 interchanged. The variations in mode shapes described thus far can be attributed to material differences. Figures 2 and 3 can be compared and the first three mode shapes are similar for $L/a = 1$ and 2. However, the second mode shape of Figure 3 occurs as the fourth mode shape of Figure 4, $L/a = 4$. Figures 2–4 all represent the same material and the variation in longitudinal/radial mode shapes between figures can be attributed to geometry.

The analysis of Table 2 for $m = 0$ and pure torsional frequencies indicates that the first torsional mode in every case is dominated by the ε_{rz} strain. The frequency is a multiple of $\pi/2$, $\pi/4$ and $\pi/8$ for $L/a = 1$, 2 and 4 respectively. Additionally, a study of Table 2 for $L/a = 1$ indicates that the first, second, fifth and ninth torsional modes are multiples of $\pi/2$. The first, second, third, fifth and eighth frequencies are multiples of $\pi/4$ for the isotropic case $L/a = 2$. The first seven torsional frequencies of Table 2 for isotropic materials that correspond to $L/a = 4$ and $m = 0$ are multiples of $\pi/8$ and serve to illustrate the effect of geometry, that is, as the length increases the effect of the shear strain $\varepsilon_{\theta z}$ is more dominant. Also, the accuracy of the finite element analysis can be assessed independently of reference [1]

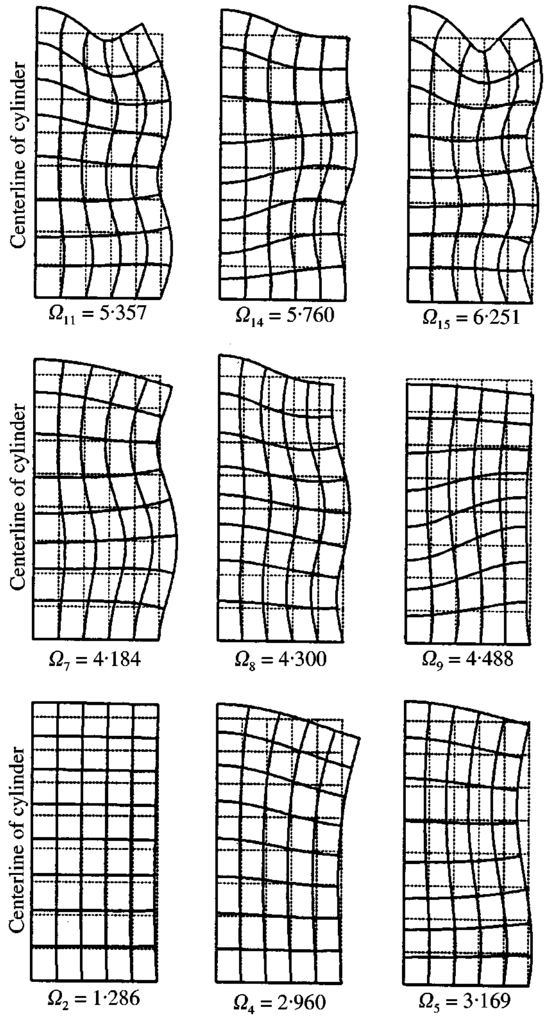

Figure 3. Mode shapes for longitudinal/radial motion of fixed–free cylinders with $L/a = 2$ and isotropic material with $\nu = 0{\cdot}3$ and $m = 0$ corresponding to Table 2.

and the seventh torsional frequency for $L/a = 4$ was computed as 5·153 and compares with $13\pi/8$ (5·105) to within less than 1 per cent. Also, the fifth frequency given in reference [1] and shown in Table 2 as 3·544 compares with the exact value of $9\pi/8$ (3·5349) to within less than 0·5 per cent and it follows that the results given in reference [1] are quite accurate.

The underlined single digit numbers in Table 2 represent an analysis of the numerical order of the frequencies and can be used to assist in assessing the effect of material versus geometry on the frequency of vibration. The lowest (first) frequency corresponds to $m = 1$ and the first mode for all fixed–free cylinder lengths and materials. Similarly, the first torsional mode occurs as the second frequency in all cases. The effect of geometry appears to dominate material effects for $L/a = 1$ since the third frequency corresponds to the first $m = 2$ frequency for both materials. However, as L/a increases, material characteristics tend to affect the order of the frequencies.

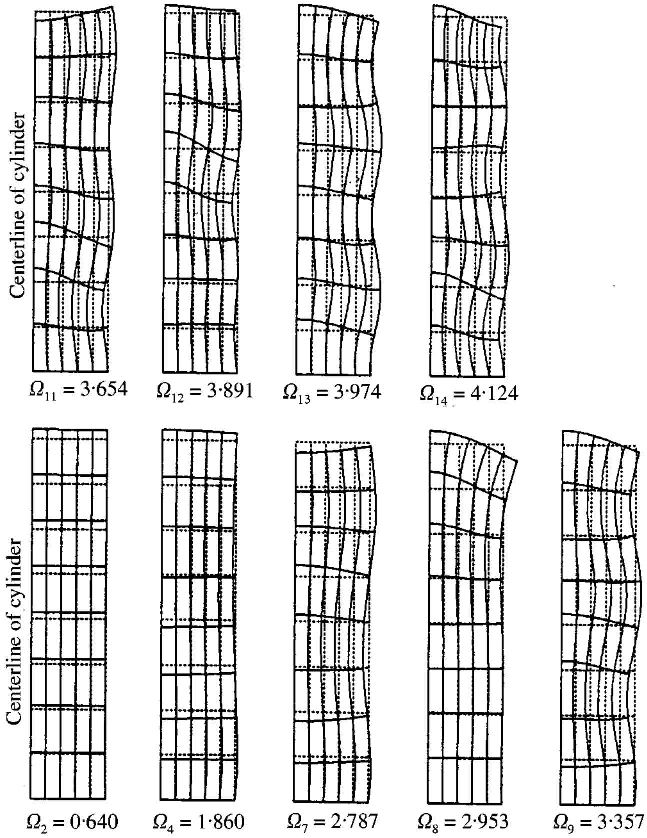

Figure 4. Mode shapes for longitudinal/radial motion of fixed–free cylinders with $L/a = 4$ and isotropic material with $\nu = 0{\cdot}3$ and $m = 0$ corresponding to Table 2.

Frequencies for fixed–fixed solid cylinders are given in Table 3. Selected mode shapes are shown in Figures 5–7. Symmetrical boundary conditions, such as fixed–fixed, lead to mode shapes that are symmetrical or antisymmetrical with respect to the center of the cylinder. The symmetries of longitudinal/radial modes for $m = 0$ are identified in Table 3 and are illustrated in the figures. Observations similar to those for fixed–free cylinders are applicable for fixed–fixed cylinders. Figure 5 corresponds to isotropic cylinders but when compared with beryllium cylinders [12] the second and third, fifth and sixth, and eighth and ninth mode shapes are interchanged respectively. Mode shapes for beryllium cylinders are shown in Figure 6 and when compared with the isotropic counterparts only the fifth and sixth mode shapes are interchanged. Mode shapes for beryllium cylinders are shown in Figure 7 and appear in the order of 1, 2, 4, 3, 6, 5, 9, 7, 8 when compared to isotropic cylinders of the same length. It is possible, with enough information, to delineate and separate the effects caused by geometry from those caused by material properties.

The order of the first nine frequencies is shown in Table 3 using an underlined number that appears to the right of each column of frequencies. The order of the first three modes is the same for fixed–free and fixed–fixed cylinders as shown by comparing Tables 2 and 3. However, fixed–fixed cylinders vary from fixed–free in the order of the frequencies as evidenced by frequencies corresponding to $m = 4$ occurring for fixed–fixed boundary

TABLE 3

Frequencies Ω for solid cylinders with fixed–fixed boundary conditions

		L/a					
		Isotropic $\nu = 0{\cdot}3$			Beryllium		
m	mode	1	2	4	1	2	4
0	1	4·671a 6	2·582s 3	1·288s 4	3·666a 4	2·257s 3	1·128s 3
	2	5·243s 7	3·749a	2·445a	4·516s 7	2·740a 6	2·252a 9
	3	5·904s	3·956s	3·178s	5·256s	3·262s	2·547a
	4	7·271s	4·822a	3·614a	6·353s	4·306a	2·603s
	5	8·408s	5·021s	3·720s	7·424a	4·529a	2·841s
	6	8·786a	5·625a	3·836a	7·965s	4·546s	3·202s
	7	9·084a	6·228s	4·282a	8·291a	5·634s	3·402s
	8	10·66s	6·695a	4·290s	9·043a	5·887a	3·714a
	9	10·873	7·225	4·498	9·245s	6·772s	4·028s
0*	1	3·142 2	1·571 2	0·785 2	3·142 2	1·571 2	0·785 2
	2	6·022	3·142 6	1·571 5	5·606	3·142 9	1·571 5
	3	6·285	4·718	2·359 8	6·285	4·718	2·359
	4	8·117	5·372	3·153	7·814	4·902	3·153
	5	9·009	6·022	3·961	8·245	5·607	3·961
	6	9·437	6·307	4·790	9·436	6·307	4·710
	7	10·526	6·975	5·197	9·886	6·620	4·790
	8	10·744	7·922	5·372	10·517	7·792	4·902
	9	12·170	8·134	5·645	11·082	7·832	5·209
1	1	3·012 1	1·433 1	0·627 1	2·953 1	1·387 1	0·589 1
	2	4·194 4	2·667 4	1·250 2	3·851 5	2·490 5	1·189 4
	3	5·342 9	3·196 7	2·001 6	4·721 9	2·771 7	1·898 6
	4	5·917	3·296 8	2·264 7	5·408	3·177	2·199 8
	5	6·765	4·230	2·811	5·652	3·837	2·435
	6	6·925	4·355	2·857	6·564	4·094	2·680
	7	6·964	4·884	3·139	6·783	4·620	2·770
	8	7·512	5·529	3·198	6·852	4·991	2·959
	9	8·746	5·611	3·593	7·651	5·064	3·263
2	1	3·734 3	2·684 5	2·383 9	3·510 3	2·438 4	2·143 7
	2	5·333 8	3·309 9	2·529	4·708 8	3·085 8	2·292
	3	5·533	4·078	2·883	5·073	3·840	2·634
	4	6·359	4·454	3·251	6·086	3·931	3·075
	5	7·727	4·761	3·347	6·825	4·352	3·330
	6	7·989	5·364	3·790	7·240	4·737	3·537
	7	8·187	5·537	4·091	7·646	5·304	3·721
	8	8·906	6·148	4·318	8·097	5·742	3·845
	9	9·124	6·622	4·487	8·475	6·051	3·904
3	1	4·601 5	3·821	3·628	4·216	3·419	3·240
	2	5·982	4·232	3·712	5·593	3·893	3·328
	3	6·637	4·934	3·928	5·751	4·769	3·538
	4	6·938	5·201	4·227	6·607	4·773	3·865
	5	8·693	5·868	4·426	8·031	5·127	4·300
	6	9·088	6·359	4·647	8·083	5·707	4·334
	7	9·202	6·479	4·879	8·638	5·997	4·637
	8	9·327	6·790	5·159	8·690	6·173	4·790
	9	10·044	7·599	5·427	9·301	6·921	5·025
4	1	5·498	4·875	4·734	4·960	4·326	4·196
	2	6·634	5·221	4·805	6·243	4·737	4·274
	3	7·587	5·844	4·966	6·889	5·467	4·437
	4	7·976	5·984	5·213	7·198	5·685	4·694
	5	9·673	6·758	5·454	8·926	6·242	5·050
	6	9·759	7·050	5·555	8·964	6·450	5·385
	7	10·020	7·724	5·789	9·218	6·887	5·513
	8	10·208	7·879	6·002	9·789	6·993	5·597
	9	11·206	8·343	6·230	10·020	7·630	5·903

Note: *—torsional mode; s—symmetrical mode; a—antisymmetrical mode.

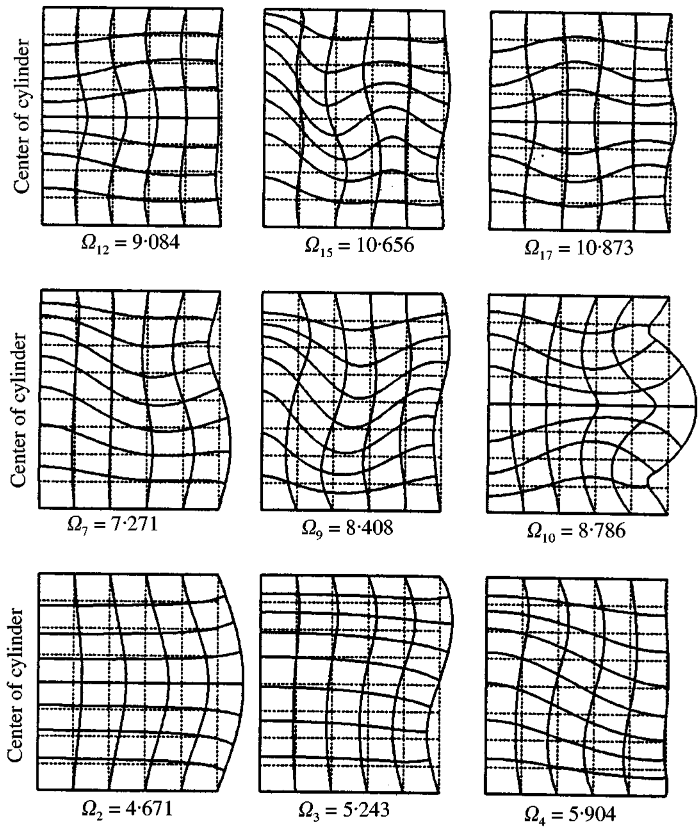

Figure 5. Mode shapes for longitudinal/radial motion of fixed–fixed cylinders with $L/a = 1$ and isotropic material with $\nu = 0{\cdot}3$ and $m = 0$ corresponding to Table 3.

conditions when $L/a = 1$. It does appear that as the cylinder length increases, $L/a = 4$, the ordering of the frequencies becomes similar.

A comment is in order concerning modelling a completely fixed boundary using numerical methods and the equations of three-dimensional elasticity. In general, to achieve a fixed condition the boundary conditions should be specified to avoid rigid body displacements. In this analysis the axisymmetric formulation would make $U(r = 0)$ equal zero without actually specifying $U(r = 0)$ as a boundary condition. So [11] has demonstrated, in a limited but correct manner, that the results of Table 2 for fixed–free boundary conditions are correct, by making a comparison with existing theories. In the absence of published results based upon three-dimensional elasticity, So [11] made the comparison with one-dimensional Timoshenko beam frequencies for flexural ($n = 1$) modes and the agreement was within approximately one per cent for $L/a = 6$. The axisymmetric finite element formulation is apparently over-constrained if $U(r = 0)$ is specified as a boundary condition. Additionally, Table 4 can be used to validate the finite element results. Assume a hollow cylinder with inside radius b. Table 4 shows the frequency for $L/a = 2{\cdot}0$ as b approaches zero. The frequencies for $b/a = 0{\cdot}001$ are identical with the results of Tables 2 and 3. Additionally, singularities such as the $1/r$ condition as r approaches zero are avoided when using Gauss–Legendre quadratures for numerical integration. The integration points always lie within the boundaries of the element and r can never become numerically zero.

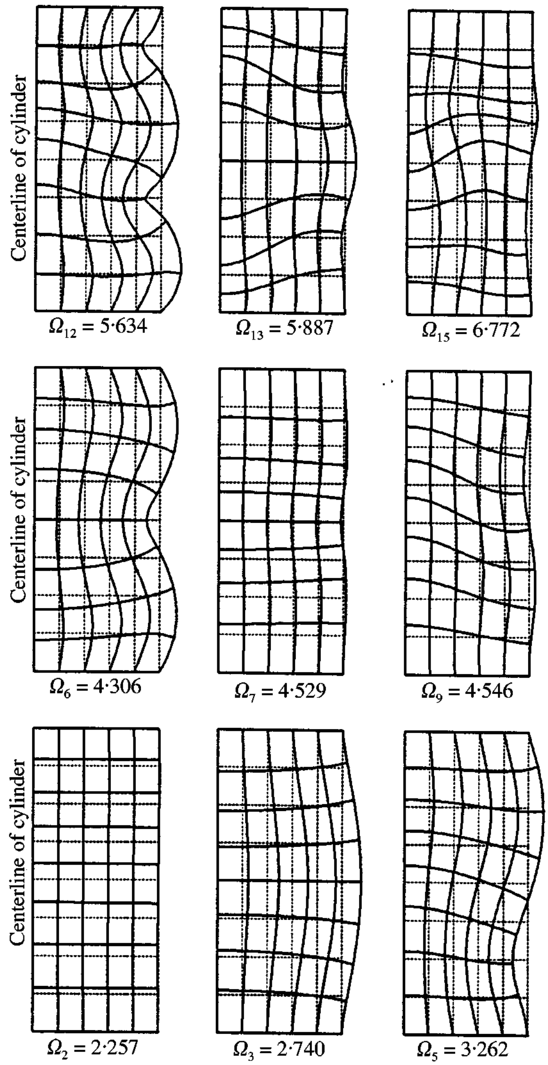

Figure 6. Mode shapes for longitudinal/radial motion of fixed–fixed cylinders with $L/a = 2$ and beryllium material properties and $m = 0$ corresponding to Table 3.

4. CONCLUDING REMARKS

Frequencies and modes shapes have been reported for a number of finite cylinder length to radius ratios and two different materials with axisymmetric material properties. It was demonstrated that the finite element formulation gave sufficiently accurate results for frequency of vibration. Boundary conditions for fixed–free and fixed–fixed cylinders were included as primary examples to augment and extend the results in literature. The analysis could be extended to include a greater variety of L/a ratios and additional hexagonal materials can be studied. If follows that the finite element analysis is quite versatile and can be applied in numerous practical situations.

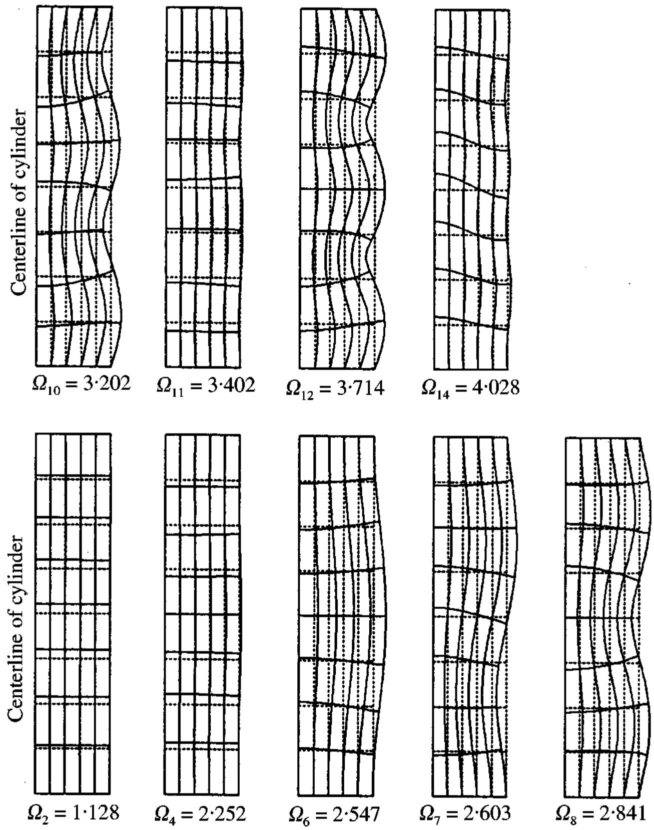

Figure 7. Mode shapes for longitudinal/radial motion of fixed–fixed cylinders with $L/a = 4$ and beryllium material properties and $m = 0$ corresponding to Table 3.

TABLE 4

Convergence of frequencies Ω for hollow isotropic cylinders with inside radius b/a, $\nu = 0{\cdot}3$ and length $L/a = 2{\cdot}0$

	b/a			
Mode	0·001	0·01	0·05	0·1
Fixed–free cylinder				
1	1·286	1·286	1·286	1·285
2	2·960	2·960	2·951	2·971
3	3·169	3·168	3·158	3·124
4	4·183	4·182	4·158	4·024
5	4·298	4·298	4·298	4·287
Fixed–fixed cylinder				
1	2·582	2·582	2·581	2·580
2	3·749	3·748	3·723	3·639
3	3·957	3·956	3·949	3·920
4	4·822	4·821	4·804	4·750
5	5·021	5·021	5·030	5·052

REFERENCES

1. A. W. LEISSA and J. SO 1995 *Journal of the Acoustical Society of America* **98**, 2136–2141. Accurate vibration frequencies of circular cylinders from three-dimensional analysis.
2. J. SO and A. W. LEISSA 1997 *Journal of Vibration and Acoustics* **119**, 89–95. Free vibrations of thick hollow cylinders from three-dimensional analysis.
3. J. R. HUTCHINSON 1967 *Journal of the Acoustical Society of America* **42**, 398–402. Axisymmetric vibrations of a solid elastic cylinder encased in a rigid container.
4. E. DIEULESAINT and D. ROYER 1980 *Elastic Waves in Solids*. New York: John Wiley.
5. R. G. PAYTON 1983 *Elastic Wave Propagation in Transversely Isotropic Media*. The Hague: Martinus Nijhoff.
6. C. P. LUSHER and W. N. HARDY 1988 *Journal of Applied Mechanics* **55**, 855–862. Axisymmetric free vibration of a transversely isotropic finite cylindrical rod.
7. P. R. HEYLIGER 1991 *Journal of Sound and Vibration* **148**, 507–520. Axisymmetric free vibrations of finite anisotropic cylinders.
8. P. R. HEYLIGER and A. JILANI 1992 *International Journal of solids and structures* **29**, 2689–2708. The free vibrations of inhomogeneous elastic cylinders and spheres.
9. J. F. NYE 1969 *Physical Properties of Crystals*. London: Oxford University Press.
10. G. R. BUCHANAN 1995 *Theory and Problems of Finite element Analysis*. New York: Schaum's Outline Series, McGraw-Hill.
11. J. SO 1993 *Ph.D. Dissertation*, *The Ohio State University*, Columbus: Three-Dimensional Vibration Analysis of Elastic Bodies of Revolution.
12. C.-L. CHUA 2000 *M.S. Thesis*, *Tennessee Technological University*, Cookeville: Free Vibration of Finite Length Solid Circular Cylinders.